A CRITICAL APPRAISAL OF THE FORMALIST FOUNDATIONS OF MATHEMATICS

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ABSTRACT

This paper is a study of the formalist contributions to the foundations of mathematics. The objective of the essay is to critically appraise the submissions of the formalist on what constitutes the foundations of mathematical propositions. The method adopted for this purpose is content analysis, which involves the study and evaluation of texts of formalist philosophies of mathematics. The study involved an analysis of the works of four key formalists, namely Hein, Thomae, Hilbert, and Curry. The early formalists, namely Hein and Thomae, did not produce formidable theoretical analysis. Hence, the study did not dwell much on their contributions. It was the theoretical submissions of Hilbert and Curry that formed the bulk of the analysis because of their depth and contemporary relevance. Although the theses of Hilbert and Curry were found to be a product of the scepticism of the object and a subsequent fallacy of ontological convenience, their contributions to the development of meta-mathematics and consequently modern mathematical logic were well noted.

***Keywords:** Arithmetic, Meta-mathematics, Formalism, Finitism, Incompleteness, Consistency.*

Introduction

The idea of formalism is not new to logical and mathematical studies. Greek mathematics was a formalisation of Egyptian geometry. The logical processes of human thought were formalised by Aristotle in his syllogism. Thus, the deductive method and its consequent formalistic structure are what Greek science has contributed to logical and mathematical research. This is not, however, to say that deduction is not human. But it is to argue that its systematic demonstration is historically rooted in Greek science. The predominance of a formalistic orientation in Greek mathematics, especially Euclidean geometry, led to the assumption that the axiomatic method was more essential to the Euclidean project than its contents, which were the descriptions of actual space. Such an assumption led great thinkers like Leibniz to regard Greek mathematics as a demonstration of logical perfection. Thus, he is quoted as having argued that "the Greeks reasoned with the greatest possible justice in mathematics" (Blanche, 1965, p.1). "In making it compulsory for schoolchildren, the aim is not so much to teach them truths but rather to discipline the mind, its practises being reputed to promote and develop the habit of rigorous reasoning" (Blanche, 1965, p. 2). On such a basis, Brunschvig has argued that Euclid, for the numerous generations who have been brought up on his book, has been less a teacher of geometry than a teacher of logic.

The foregoing assumption with respect to Euclidean geometry was made because it was thought that the three-fold formalistic features of formal systems were their essence. Such features are (i) the undefined terms; (ii) the undemonstrated propositions put forward as hypotheses; and (iii) the other propositions constructed from them according to the rules of logic explicitly stated. But unfortunately, Euclidean geometry has in recent times been exposed as a pretence at sheer logical rigor. At least one of its postulates has really faced criticism due to an unfounded assumption. The postulate is his parallel postulate. It states that given a straight line, there is a point such that one and only one straight line can be drawn through it. This postulate has been shown to be logically untenable. The lines passing through the point define an infinite possibility. Besides, Euclid's system was supposed to be purely deductive. Its axioms were assumed to be general rational principles. The postulates were to be understood as descriptions of space, on the basis of the axioms and the theorems to be deduced from these postulates. But Euclidean geometry is a complete deviation from these expectations. Mathematicians get so frustrated with Euclid's inductive statements, which possess non-deductive properties. In short, properly described, Euclidean geometry is a description of intuitively understood actual space.

The logical inadequacies of Euclidean geometry resulted in the formation of new and more rigorous geometrical systems called non-Euclidean geometries. The beauty of non-Euclidean geometries is in their logical rigor. Consequently, non-Euclidean systems possess, as a property, the summation ($\sum$) of Euclidean postulates minus, the parallel postulate (A). Thus, a non-Euclidean geometry is of the form ($\sum -A$). Where $\sum$ represents all other Euclidean geometric postulates and (A) the postulate of parallel (Wilder, 1955 p.29). An important aspect of non-Euclidean geometry is that the subtraction of (A) is not the same as its contradiction. It is rather a replacement with some other postulate, which is necessarily contrary to it but not contradictory, such that a new system emerges. There is, therefore, no end to possible geometrical contraries to the parallel postulate. The logical consistency of the new geometry tends to awaken the logical ideal expected from the Euclidean system. The primacy of logicality undermines the importance of contents, thus giving rise to the analysis of these systems in terms of their combinatorial structures. This then became the impetus for formalism in modern mathematics and logic. The consequence of this for Euclidean geometry was the subtraction of its postulate and the demonstration of a logically consistent Euclidean geometry without the postulate of parallels by David Hilbert.

The Notion of Formalism in Mathematics and Logic

Symbolic absolutism in mathematical analysis is the essential characteristic of formalism. Thus, according to Hamilton (1978):

> *The word "formal" … is used when referring to a situation where symbols are being used and where the behavior and properties of the symbols are determined completely by a given set of rules. In a formal system, the symbols have no meanings, and in dealing with them we must be careful to assume nothing about their properties other than what is specified in the system (p.27).*

Despite this explanation, it is noteworthy that the assumption of nothing in the construction of the formal system has been intuitively provided for in syntax. The nature of what is studied determines the rules. Thus, there is no possibility of formalisation without intuitive assumptions. This argument will become evident as the analysis proceeds.

Thus, ordinarily all formal systems possess the following properties:

(1) An alphabet of symbols.

(2) A set of finite strings of these symbols, called well-formed formulas. These are to be thought of as the words and sentence in our formal language.

(3) A set of well-formed formulas, called axioms

(4) A finite set of 'rules of deduction', i.e., rules which enable one to deduce a well-formed formula A, say, as a 'direct consequence' of a finite set of well-formed formulas A_1, … A_k, say (Hamilton, 1978, p.27-28).

All systems constructed in this way are assumed to be consistent, and their model is sure. After all, there is no formal construction without presuppositions. The presuppositions determine the rules.

Hilbert assumed this kind of system yet rejected presuppositions and based it on arbitrariness in his attempt to establish the consistency of mathematics and the reality of its entities. As Frege has pointed out, if no ontological assumptions or presuppositions are made prior to the construction of the system, then the exercise is epistemologically futile. Thus, it is impossible not to presuppose an intuitive basis in formalistic constructions if their logical properties are to be considered. The history of mathematics reveals this in the analysis of geometrical formalism. Contemporary geometry, reduced to arithmetic and logic, makes unexpressed logical and arithmetic intuitive assumptions in its constructions. But it is nonetheless painful that the formalists in the foundations of mathematics are men with inconsistent claims. Claiming the independence of mathematical language, the formalists smuggle in the idea of an objectively existing combinational linguistic realm as the basis of formalization. Thus, in spite of its pretensions, traditional formalism is guilty of the fallacy of ontological convenience due to its assumptions of linguistic conceptualism, esoterism or empiricism as the foundation of mathematical formalism or constructions. Thus, "formalism is the identification of the sources of mathematical knowledge in mathematical symbolism or formalism" (Resnik, 1980, p. 54). What formalists seek to demonstrate is a meaningless language system or a completely non-objective ontology outside of symbols in mathematical practice. Hilbert had argued that "in the beginning there was a sign" (p. 82).

The implication of the above analysis is that the seeds of contemporary axiomatic method are found in Greek and modern mathematical analysis. The tendency towards a first rigorous axiomatization of geometry was first found in Pascal in 1882. His intention was to provide the logical strictness claimed for Euclidean geometry. Even at that, axiomatic geometry presupposed the intuition of real and actual space as what is formalized. Thus, the view of the practise as mere symbolic, meaningless combinations is actually identifiable with a contemporary philosophy of mathematics called formalism. The frustrating results in meta-mathematics are an indication of the philosophical mistakes of the system. Meaningless signs and symbols are inadequate; signs are, in reality, transcendental. They are not the realities intended in their usage. But the formalists have, on the contrary, settled down for signs.

The Formalist Foundations of Mathematics

Contentment with signs, as all there is for mathematical demonstration, has its roots in the idea of complex numbers, introduced by Bombelli, for providing solutions to hitherto unsolvable mathematical problems. The signs of Bombelli had no transcendence. They were the numbers themselves. Mathematicians used these numbers without asking for their meanings (Resnik, 1978, p. 56). Gauss used the notion in a geometrical demonstration, and from that time on, mathematicians came to view numbers and, in short, all of analysis in terms of geometry. This view of numbers prevailed until Weiestrass, Dedekind, and Cantor arithmetized analysis. The arithmetization of analysis introduced the notion of actual infinity.

This notion was vehemently opposed by Kronecker, whose confidence was threatened. Mathematics had been content with operations within known instances. How can mathematicians understand the notion of actual infinity? For some mathematicians, the project of naïve set theory posited by Cantor and Dedekind was problematic because of the notion of transfinite numbers and the attendant paradoxes it created in set theory. Set theory had to be axiomatized in order to address the problem of paradoxes. The axiomatization of set theory was the expurgation of meaning. Some would pretend that logical constants were to retain their meaning. But it is clear that they do not understand the implication of the new opening. The retention of logical constants and their meaning in formalised set theory marks an introduction of the intuition of being in general into a supposed formal combinatorics. This results in a contradiction. There is at once the claim to meaninglessness and then the claim to meaningfulness. If this contradiction is removed, then the formalism of set theory is made consistent. Gödel has demonstrated that such a system is mathematically incomplete for the proof of its logical consistency. The first attempt at addressing the problem of actual infinity collapses.

Kronecker claims, therefore, that we cannot intuitively talk about actual infinity and be correct. According to him, we can only accept potential infinity. What was at stake was the problem of mathematical existence or the foundation of mathematical propositions, especially non-finite propositions. The problem faced is one that philosophers have tried to resolve down the ages. The main problem is how our science of mathematics is possible. This question has received extensive attention in Kantian philosophy. It is also not impossible to observe the Kantian influence running through some of the projects in the foundations of mathematics. One of the main complications of Kantian philosophy was his inability to transcend appearances to the things-in-themselves. The difficulty in Kantian transcendence is immediately expressed in the formalist response(s) to the problem of the cause(s) of mathematical propositions or the foundations of mathematical claims, especially concerning the existence of infinite and transfinite numbers. Regarding the question of actual infinity, Hein, a member of the formalist school, responded by arguing that "… the symbol and the number are one and the same" (p. 56). Accordingly, suppose that I am not satisfied to have nothing but positive rational numbers. I do not answer the question "What is a number?" by defining numbers conceptually, say, by introducing irrationals as limits, whose existence is presupposed. I define from the standpoint of a pure formalist and call certain tangible signs numbers. Thus, the existence of these numbers is not in question (p. 55).

Hein never understood the depth of the implications arising from his claims. So many signs are numbers. Any sign can be a number, provided I decide that they are numbers. In the same vein, even the arithmetic numeral could be denied the quality of being a number if I so decided. Assuming this was granted to Hein, it would be impossible to understand how mere signs possess mathematical properties. If numbers are sheer signs, then chemical properties of signs are mathematical properties of numbers. But it does not follow. Hein spoke with his kind of confidence because he felt, that mathematics was a social convention. But he never observed the overwhelming uniformity of the nature of mathematical operations in completely diverse cultures. If he did, then he would have considered mathematical operations to be fundamentally human. Such generality, if granted, would result in linguistic esoterism, idealism, or platonism. To avoid such Platonism as the foundation of Hein’s formation is to completely misunderstand him. The reason is that he assumes the uniformity of mathematical operations by human beings. Besides, he knows very well that it is an empirical fact.

One of the reasons for the importance of Hein in the history of the foundations of mathematics lies in the fact that his response marked the origin of formalism in the field of research. Thus, Hein sought to return to the Bombellian formalist safe havens. But Frege and other mathematicians have sought to make it clear that mathematics is not a meaningless combinatorial linguistic system. It is a science. So, Hein must be forced out of the havens to face the facts.

The strand of formalism championed by Hein is called game formalism. A more systematic demonstration of this system was carried out by Thomae. On the whole, formalism in the foundations of mathematics divides historically into three brands, namely:

(1) Game formalism, which takes mathematics to be a meaningless, chess like game in which the symbolism functions as the 'board and pieces',

(2) Theory formalism, which treats mathematics as the theory of formal systems;

(3) Finitism, which views part of mathematics as a meaningful theory of certain symbolic objects and the remainder as an instrumentalistic extension of the former (p.54).

Game formalism is traceable to Hein and Thomae. Theory formalism is identifiable with Haskell Curry and finitism with David Hilbert. The criticisms that rained on Hein almost undermined confidence in formalism. But Thomae sought to restore the formalist position by arguing for a more systematic formalism. Thus, he argued, arithmetic is like a chess game. The symbols represent the pieces. The symbols are meaningless on their own. Their operations depend on the rules that assign behaviour to them (p. 56). What is implied here is that arithmetic is actually found in the specified rules. It leads to arbitrariness and the proliferation of rules and arithmetic systems. In that sense, anything can parade as arithmetic, so long as there are symbols and rules. Chess games, draft games, scrabble, etc., are all arithmetic. The implications opened in Thomae's analysis are controlled by his argument, that the rules are designed in such a way that the resultant axioms capture the perceptual manifold (p. 56). What this means is that the basis of the rules of the game are abstracted regularities of the arrangements of the perceptual manifolds. So, arithmetic is founded on manifolds. This goes back to Mill's empiricism (Mill 1950). This empiricism was responsible for the kind of mathematical analysis presented by Thomae.

He argued that the whole of arithmetic "as a computing game is constructible in the familiar way from the signs 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 …" (p.57). But as Frege has pointed out, Thomae used more signs than he listed, and gave no indication of how the additional signs are composed from the 10 signs given (p.57). But Thomae argued that such signs should be understood in the familiar way (p.57). The idea of the "familiar way" has a bi-polar interpretation. The "familiar way" as Frege claims is a presupposition of meaningful arithmetic. In that sense Thomae would be guilty of the fallacy of *petitio principii*. But if the "familiar way" is understood in terms of the familiar arrangements of the perceptual manifolds, then Thomae's formalism becomes inconsistent. In either case, his formalism would be problematic. Thomae actually presupposed meaningful arithmetic. For instance, he used the commutative law without first providing for it in his system (p.58). Again, the law could be based on the arrangements of the perceptual manifolds.

In short, inconsistency is what Thomae cannot avoid in his system. He assumed that arithmetic is a chess-like game of meaningless symbols. Later he argued that the axioms should be arranged in such a way that they capture the perceptual manifolds. Why the perceptual manifolds? What Thomas's system is demonstrating is the indispensability of intuition in mathematical construction. Properly understood, intuition is knowledge produced as the cooperation of the subject and object. A lot go into the construction of a formal system. One of such is natural logic, which is fundamentally intuitive or naïve. Such logic and the object of its operation are inexpressible in the formal alphabets. Thus, it is in the interpretation of the system by virtue of its specified domain that assumptions of the system become evident. Such interpretation is the formalistic syntax. The syntax of a game is determined by its intension. So, what is the intension of mathematics for Thomae? It is the comprehension of the perceptual manifolds in mathematical formalism. Thus, what was supposed to be meaningless has become meaningful, by the smuggling in of what was denied; intuition.

The foundation of formalist's problems is the assumption of two incompatible properties of axiomatization, namely; consistent formalism and completeness theorems. But as Gödel has shown in his meta-mathematical results, no consistent formalism is complete for the demonstration of consistency proof. Such formalism marks the expulsion of the schema of comprehension expressed set-theoretically as thus, if $\varphi$is a property, then there exists a set Y = {X: φ(x)}. Russell argues with other mathematicians that this schema is responsible for the paradox of set theory. So, the trend is to demonstrate a set theory without meaning. But Gödel has shown that all such set theories are mathematically incomplete. Incompleteness is the absence of intuition. And as Thomas Jech (1987) has argued, in set theory it is not the schema of comprehension that is paradoxical but the idea of the set of all sets (p.2). It is a distorted ontological assumption, with respect to sets, that are responsible for the paradox. Such assumption rest on sheer ontological convenience and not discovery. Anyone, who carefully studies the theory of logical types of Bertrand Russell, would discover that it is based not on theoretical necessity but academic policy. In the hierarchy of types, it has been decided that in the consideration of a lower type, a higher type is not included. That is the consequence of problematic formalistic assumptions.

David Hilbert had argued that the problem of existence and truth could be addressed from the viewpoint of consistency proof. Consistency was the main thrust of Hilbert's formalism. Yet he had to face the problem of existence and truth equally. The early Hilbert had thought, that if the consistency of a system is proven, then the truth of it propositions and the existence of its entities are sure. But the whole project of Hilbert is set forth in two inconsistent schemata. The idea of consistent formalism is inconsistent with the demand of logical consistency of the resultant system. One is non-intuitive whereas the other is intuitive respectively. One is absolutely meaningless whereas the other is meaningful, respectively. So, it is an introduction of problem, in short much problem, to seek the consistency proof of a strictly formal system. Gödel showed that it is impossible. The system is not complete to be able to carry out the project. Consistency proof is intuitive whereas formalism is not. Some have argued that the problem is resolvable if the axiom of consistency were introduced into the system. Such axioms like: p and ¬p. But Gödel has shown that if that concession were made at all, then there would be at least one statement in the system, which does not belong to it. Thereby making the system formalistically inconsistent. Thus, it is only when the system is inconsistent formalistically, that it is complete. But that is not meant to identify completeness with inconsistency. Formalistic inconsistency is a function of formalistic claims. If one were to claim that a system is strictly formalistic and yet demonstrates one that is actually axiomatically intuitive possessing statements that are not derivable from its alphabets, then the person, in question, could be said to be guilty of formalistic inconsistency, which is inconsistency in claims and demonstration.

Consequently, strict formalism is incomplete for the proof of logical or formalistic consistency. Strict formalism was what David Hilbert assumed and his intension was the proof of the logical consistency of mathematics. From the above analysis, the mathematical consequences of the programme are indubitable. Hilbert's was incomplete. His project could not be carried out. Besides the idea of non-finite mathematics, which he sought to show, to be consistent with finite mathematics was intuitive and as such inexpressible in his meta-mathematics. What frustration could be greater than that? Hilberts had to face the frustrating results arising from the works of the young Austrian mathematician, Kurt Gödel. He had argued with and ignored Frege's and Kronecker's criticisms. But he could not ignore the mathematical results of the young Gödel.

David Hilbert, a German mathematician got into this trouble in his attempt to free what they call non-finitary mathematical statements from the attack of critics. Hilbert had argued that no one can send us out of the paradise which Cantor had created for us (Korner, 1971 p.73). But the paradise gave rise to the paradoxes. Hilbert sought to resolve the paradoxes by providing the consistency of all of mathematics. His method of handling the project was what deprived him of Cantor's paradise. Passion for truth especially when the legitimacy of transfinite inductive methodology has been shown cannot allow for the notion of the infinite to be abandoned. Yet to keep it, the idea and its referent would have to be properly contextualized and consistency demonstrated in complete or modeled systems.

The contextualization of mathematical truths is what Hilbert has achieved in his theoretical analysis. Mathematics he argues has finite and definite objects (Resnik, 1980 p.80). These objects are the unary numeral and its production rules (Korner, 1971 p.77). The numeral is the figure "1" (Korner, 1971 p.77). In this sense, Hilbert followed Kant's but not Leibniz's idea of mathematical propositions. Leibniz had thought of mathematical propositions as sheer expressions of logical forms. But the role of logic in mathematics according to Kant, is not different from its role in other disciplines (Korner, 1971 p.72). Thus, mathematics is not logic, in the sense of being the demonstration of logical principles. The self-evidence of mathematical propositions is founded on their description of real existences. Kant thought of these existences primarily as the a priori intuitions of space and time (Korner, 1971 p.72). Hilbert is quoted by Korner as supporting this position in the following way:

> *… something which is presupposed in the making of logical inferences and in carrying out of logical operations is already given in representation … i.e. certain extra-logical concrete objects, which are inductively present as immediate experience, and underlie all thought. If logical thinking is to be secure, these objects must be capable of being exhaustively surveyed in their parts; and the exhibition, the distinction, the succession of their parts, and their arrangement beside each other, must be given, with the objects themselves as something that cannot be reduced to anything else or indeed be in any need of such reduction (p.73).*

These irreducible for arithmetic are the unary numeral and the production rules; the figure '1' and rules of generation of other figures.

Previously, Hilbert had sought the proof of consistency in logical implication of concepts. But in the new systems, he reduces questions of consistency to questions about derivability (Resnik, 1980, p.80). After all, he had argued that in the beginning there was a sign. In this overly symbolic domain, the proof of consistency of a system is attainable only in a formalistic context. The main property of the system, the existence of which is of necessity is the system of rules, which also contains the terms, the operations, the propositions and rules of transformations or inference. The provision of the terms and the rules of grammar and inference is however adequate for the demonstration of proofs of derivability. The resultant system, which is the meta-system, in Hilbert terminology, is an expression of the formal system (Mostowski, 1964 p.26). It is a demonstration of the formalistic properties of the system. Thus, given the figure '1' and the production rule, Hilbert believes that the other natural numbers follow as thus: *1, 11, 111* etc. If these are translated into decimal numbers, we have 1, 2, 3 etc.

Hilbert is of the opinion that every other aspect of mathematics is a function of the figure '1' and its rules of production. He therefore divides mathematics into two major parts, namely; the construction and the theory. Mathematical construction is a function of rules. Theory is the description of constructions. The justification of the theory depends on the formal properties of the rules. The business of the justification of theory he divided into four parts; namely, the construction, the theory, formalism and meta-mathematics. The first two are those just explained. Formalism is a reconstruction of theory or a provision of ordinary formalistic primitive frame. The frame refers to the system of syntax. The formalization of theory is what Hilbert calls formalism. Formalism is not a total rewrite of theoretical demonstration but is an abstraction of the terms and rules guiding expressions of the theory. Thus, formalism is rather a rewrite of what Hilbert refers to as initial representation. The meta-theory, which is a rewrite is a rewrite of the theory, which is an expression of formalism.

Thus, Hilbert argued that:

> *The consideration of the concrete theory alone creates a picture in which the science of mathematics is reducible to number equations, but the science of mathematics … is in no way exhausted by number equations and is not entirely reducible to such. Yet one can assert that it is an apparatus, which its application to whole numbers must always yield correct numeral equations (Korner, 1971 p.76).*

He refuted Brouwer and argued that "the formula game that Brouwer deprecates has, beside its mathematical value, an important general philosophical significance. For this formula game is carried out according to certain rules, in which the technique of our thinking is expressed" (Resnik, 1980 p.91). The said apparatus is not far from us according to Hilbert. It is the same logical thinking according to which number theory is possible (Korner, 1971 p.76). It was on the basis of this that his younger contemporary Haskell Curry declared that a formal system is a machine for the production of formulae. But it is noteworthy that Curry's idea of the apparatus has nothing to do with logicality. In short, he accepts inconsistencies in Meta-mathematics, so far as it is pragmatic.

Critical Appraisal of the Formalist Foundations of Mathematics

The implication of this for Hilbert is that mathematical theory and in short meta-mathematical demonstrations are expressions of the propositional implications of the formal system. Hilbert wished that meta-mathematics be consistent. But for that to happen, formalism would first have to be consistent. The consistency of meta-mathematics is however the consistency of formalism. But it has been discovered that meta-mathematics which is a resultant system of formalism is meaningless or non-intuitive. Consequently, the absence of intuition or meaning or internal interpretation (to use Dummett's expression) makes meta-mathematics incomplete to carry out the load of mathematical theory, which includes properties in intuitive formalistic system.

The idea of meaningful mathematics to which Hilbert appealed is inadequate. The reason is that the objects of the system are symbols, which are in themselves meaningless. Epistemologically, Hilbert's system is guilty of symbolic Platonism. Thus, the problematic in Hilbert's meta-mathematics is actually rooted in the problematic of traditional epistemology. What flung Hilbert into questionable assumption is the absence of mathematical entities in the world. Korner saw this problem very clearly, then he wrote the minutes that "formalists think that statements of pure mathematics are empirical" (Korner, 1971 p.98).

The description of Korner is ad rem with the business of the formalist. The younger contemporary of Hilbert, Haskell Curry makes the matter quite clear. Hilbert's claim of the provability of the logical consistency of the system of consistent formalism was what Haskell Curry avoided. He accepted inconsistent formalism. Logic became to him, unimportant for mathematical thinking. There is no sense of logic outside the sense in which they are found in systems of logics created by human beings. His importance in this analysis lies in the ontological and epistemological implications he drew from previous mathematical philosophies to build his system. He got these especially a'la Hilbert. Curry translated mathematics into a meta-mathematics and thus built a physics of strokes expressions. Consequently, he argued that "mathematics is a science of formal systems" (Resnik, 1980 p.65). Statements of mathematics are such as "such and such is a theorem of such and such a formal system" (p.65). A formal system he argued can be identified as containing primitive frame, which consists of presenting (1) a list of terms of the formal system (2) the elementary propositions of the formal system, and (3) the elementary theorems of the formal system (p.65).

The terms of the formal system consist of tokens, operations and rules for forming terms using the operations' (p.66). The elementary propositions are formed from elementary predicates and number and the kinds of arguments that the predicates takes (p.66). Predicates here could be understood in terms of relations, rules or functions. For instance, the primitive predicate of identity. The next thing is the determination of axioms by presenting elementary propositions under the label axioms. This is followed by rules of procedures, according to which it would be determined whether a given elementary proposition follows from another or not (p.66). "The elementary theorems of a formal system are simply the axioms and the elementary propositions that can be generated from them according to the rules of procedure" (p.66).

Curry thinks that in this way, he had achieved meaningful mathematics. But this is not far from what Hilbert did. The demonstration of Hilbert was supposed to be the demonstration of formal properties. In short finite mathematics need not be abandoned. Non-finite mathematics is the amplification of infinite mathematics. Thus, what is needed was the proof of the consistency of the system thus amplified. Hilbert thought that the finite formalistic method was the only one available. He believed that there was no more to mathematics than the concrete objects of constructions. To proof the consistency of the whole system is to show that the ideal statements are interpretable in terms of the concrete ones. Thus, in the final analysis, mathematical theory is a meta-theory of formalism or constructions. The idea of truth in both Hilbert and Curry is uniform; theoremhood.

> *In the theoremhood idea of truth, the … predicate P of a formal system (S) is defined by the condition that ordered n-tuple of terms (t_1, …, t_n) belongs to the extension of a predicate P just in case the elementary proposition Pt_1, …, t_n is a theorem in S. The elementary proposition Pt_1, …, t_n is true, however just in case (t_1, …, t_n) belongs to the extension of P (Resnik, 1980 p.67).*

Consequently, the statement ├ X is true just in case X is a theorem of a given formal system under study. This analysis is better expressed by Resnik as thus "... in ordinary formal system *ZF* (Zemelo-Fraenkel), the following sentence is a theorem: $(\forall x)(\exists y)(x \in y)$. In the Curry's correlate, … the expression'$(\forall x)(\exists y)(x \in y)$' symbolizes a term and the expression '├ $(\forall x)(\exists y)(x \in y)$' symbolizes a true elementary proposition" (p.68).

Curry opined that the statements of meta-mathematics are really meta-propositions. Meta-propositions are not meta-mathematical categorical propositions. They are rather meta-mathematical hypothetical propositions. The use of meta-proposition marks the ontological priority of the initial representation (the formal system), which Hilbert hinted on. Thus, the question that immediately arises is that of the foundation of the formal system. Hilbert and Curry cannot pretend that such a system is a generalization from the induction of systems in experience (i.e. those designed by people). If that is granted to them all, a question immediately arises concerning the cultural uniformity of mathematical formalism as a syntactical system. Korner (1971) answers this question by arguing that "if we accept the … distinctions between empirical concepts and operations and the non-empirical ones then a science of empirical stroke-expressions and operations must be distinguished from the science of idealized stroke-expression and operations. The latter alone … would be meta-mathematics" (p.105).

The above submission is alive to Hilbert and Curry. It could be recalled that Hilbert called mathematics the general apparatus, which when applied to numbers gives number equations. This was a mathematical idealism, a Platonism and a form of esoterism. Besides, Curry claimed that the property of formal systems exposed above are abstract general properties of all formal systems. The concept of generality is not the concept at issue. But what is at issue is that of the cultural uniformity of mathematical expression, which he accepts.

A much more critical appraisal of formalism, which align with the ideas of this essay came from Brouwer and Poincare. Frege had used the criticism against Hein. The accusation could be summarized as meaning that formalism presupposes meaningful mathematics. Brouwer and Poincare's arguments are put as follows (1) the formalists do not explain why we are interested in certain systems and not others; (2) they must admit contradictory results as being mathematical and (3) in setting up their system they fall back on the assumption that there is a mind independent world of mathematical objects to be described by the axioms" (p.81). The world of mathematical object for the formalist are the signs and the rules. It is formalism. The foregoing analysis shows that if formalism were accused of the cognitive autocracy of the object, then there would be no freedom. To solve the problem of the lack of object, formalism seeks a convenient ontology, in symbolism. Meaningless symbols are the domain of the formalist foundations of mathematics. Cognitive autocracy of the object is an epistemological assumption that the cognitive object in total exclusion of the input of the subject provides absolute grounds for the possibility of knowledge claims. Hence, to demonstrate the foundations of knowledge is to exclusively present the object. Where there is a skepticism of the object, such skepticism is resolved by the presentation of the existence of some queer putative domain of object as a convenient ontological domain to settle the skepticism; this exercise is termed the fallacy of ontological convenience in the paper.

The formalist mathematicians thought that meaninglessness could be avoided by meta-mathematics. But it is a pity that there is no possibility of actual meta-mathematics. Meta-mathematics is a symbolic demonstration of the formal properties of the formal system. If the formal system is already constructed as a simple demarcated system, with all its propositions set down, then meta-mathematics becomes a transformation of the formal system in the language of the system or another language. The implication of this is, that the option of meaningful mathematics is denied to Hilbert. With the denial, it becomes undecidable whether the system is consistent or not. Hence, Hilbert finitistic proof of mathematical consistency cannot be carried out.

This could tempt one to think that mathematics is inconsistent, as some persons find joy in saying. Mathematics is consistent. Hilbert's analysis of the science is one of the most ingenious analysis ever presented. He argued that the science of mathematics has both a real and an ideal part. The latter is a logical idealization of the former and it is interpretable in terms of it. In making this assertion he defeated inductive fallacy and showed the legitimacy of the universality and necessity of scientific statements. The logical idealization or induction of fact is not an idealism. It is a hypothetical analyze of a singularly truth. For instance, in a distinctly described instance, if it is discovered that '$\vdash (\exists x)\ \varphi(x)$', it could be idealized as such '$\vdash (\forall x)\varphi(x)$', without the fear of existential fallacy. The non-finitary expression of this truth could be understood as stating the following: 'if x ever has the property φ which it has in the time of judgment, then 'x is φ until it is proven otherwise'. The only mistake Hilbert made was his inability to detect that the legitimacy of this analysis is inexpressible in a non-intuitive meta-theory. Besides, Hilbert was completely ignorant of the fact that formalism would produce non-intuitive results which is not adequate for consistency proof.

Conclusion

In conclusion, what Hilbert suffers is not as a result of his analysis of the meaning of mathematics but because of his approach to the proof. If he had settled down to his analysis alone, the proof would have necessarily followed. The use of transfinite inductive analysis, which is the concept of logical idealization, suffices to prove the consistency of the system. Brouwer and Poincare made this observation and argued also that Hilbert's project of consistency proof for mathematics begs the question. Anyhow, proving the consistency of the whole of mathematics (finite and non-finite) is not the sin of Hilbert. His crime is the expurgating of intuition in an exercise that demands intuition for its legitimacy, the proof of logical consistency. This problem is rooted in traditional epistemic underestimation of the contribution of the cognitive subject in foundational analysis, which is responsible for the cognitive autocracy of the object, the objective theory of justification, the fallacy of ontological convenience, and the controversies in the programme.

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